Lecture 10

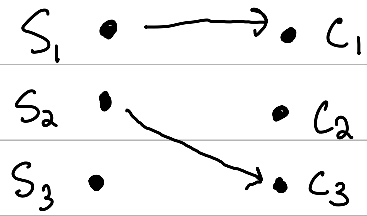
- (Binary) Relations

$A = \{s_1, s_2, s_3\}$

$B = \{c_1, c_2, c_3\}$

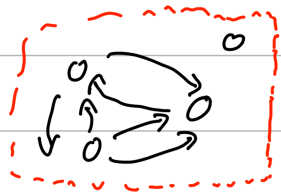
$a \in A \quad b \in B \qquad aRb \Leftrightarrow a \, c \, b$

$R = \{(s_1, c_1), (s_2, c_3)\} \subseteq A \times B$



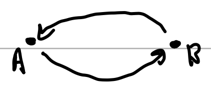
Reflexive, Symmetric, etc. as graphs

- Reflexive: i.f.f. there is a loop at every node.
- Anti-Reflexive: i.f.f. there is no loop at every node.
- Symmetric: i.f.f. for every two nodes a and b:
    - There is a directed arc from a to b
    - There is a directed arc from b to a.
    - (a and b can be distinct or not)

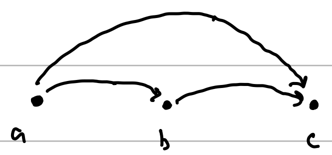


This is Symmetric.

- Antisymmetric: i.f.f. no two edges as below for distinct a's and b's.



- Transative: i.f.f. there is an arc from a-to-c if arcs from a-to-b and b-to-c.



Domain : $\mathbb{R}^+$

$\boxed{\text{Ex 1}}$ $aRb \iff \frac{a}{b} < 3$

Reflexive check: $\forall x \in \mathbb{R} : \frac{a}{a} < 3$ TRUE

Symmetric Check: $aRb \rightarrow bRa$

$\frac{a}{b} < 3 \rightarrow \frac{b}{a} < 3$

Counter example $a = 1, b = 9$

FALSE

$\boxed{\text{Ex 2}}$ $aRb \iff \frac{a}{b} > 3$

Reflexive check: $\forall x \in \mathbb{R} : \frac{a}{a} > 3$ FALSE

Anti-Reflexive check: $\forall x \in \mathbb{R} : \frac{a}{a} \leq 3$ TRUE

Symmetric check: $\frac{a}{b} > 3 \rightarrow \frac{b}{a} > 3$

Counter example $a = 4$, $b = 1$

FALSE

$aRb \iff a \rightarrow b$

Q: Is this transative

Check: $xRy \wedge yRz \rightarrow xRz$

$(x \rightarrow y) \wedge (y \rightarrow z) \rightarrow (x \rightarrow z)$

This is also reflexive because $x \rightarrow x$.

Composition of relations:

$f \circ g(x) = f(g(x))$

$aRb \quad bSc \quad$ so $\quad a(S \circ R)c$

Relations and functions are similar, but they are not equivalent.

Example

$S = \{(a,b), (a,c), (c,d), (c,a)\}$

$R = \{(b,c), (c,b), (a,d), (d,b)\}$

$S \circ R = \{(b,d), (b,a)\}$

Example

$R_1 = \{(x,y) \quad x \leq y\}$	$(x, z) \in R_1 \circ R_2$
$R_2 = \{(x,y) \quad x > y\}$	$\Leftrightarrow \exists y\,(x R_2 y \wedge y R_1 z)$
$R_1 = R_2$	$\equiv \exists y\,(x > y \wedge y \leq z)$